

Filtering hidden Markov measures

OMIROS PAPASPILIOPOULOS

ICREA and Universitat Pompeu Fabra

MATTEO RUGGIERO

University of Torino and Collegio Carlo Alberto

DARIO SPANÒ

University of Warwick

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We consider the problem of learning two families of time-evolving random measures from indirect observations. In the first model, the signal is a Fleming–Viot diffusion, which is reversible with respect to the law of a Dirichlet process, and the data is a sequence of random samples from the state at discrete times. In the second model, the signal is a Dawson–Watanabe diffusion, which is reversible with respect to the law of a gamma random measure, and the data is a sequence of Poisson point configurations whose intensity is given by the state at discrete times. A common methodology is developed to obtain the filtering distributions in a computable form, which is based on the projective properties of the signals and duality properties of their projections. The filtering distributions take the form of mixtures of Dirichlet processes and gamma random measures for each of the two families respectively, and an explicit algorithm is provided to compute the parameters of the mixtures. Hence, our results extend classic characterisations of the posterior distribution under Dirichlet process and gamma random measures priors to a dynamic framework.

Keywords: Optimal filtering, Bayesian nonparametrics, Dawson–Watanabe process, Dirichlet process, duality, Fleming–Viot process, gamma random measure.

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1 Introduction

1.1 Hidden Markov measures

A hidden Markov model (HMM) is a sequence $\{(X_{t_n}, Y_{t_n}), n \geq 1\}$, with the following ingredients: X_{t_n} is an unobserved Markov chain, called *latent signal* and assumed here to be the discrete time sampling of a continuous time Markov process; the Y_{t_n} ’s are conditionally independent observations given the signal, with law given by the *emission distribution* $\mathcal{L}(Y_{t_n}|X_{t_n})$, parametrised by the current signal state. Filtering optimally an HMM entails the sequential

exact evaluation of the *filtering distributions* $\mathcal{L}(X_{t_n}|Y_{1:n})$, that is the conditional distributions of the signal given the past and current observations $Y_{1:n} := (Y_{t_1}, \dots, Y_{t_n})$. Optimal filtering thus extends the Bayesian approach to a dynamic framework. The evaluation of the filtering distributions is the key for the solution of several statistical problems in this setting, such as the prediction of future observations, the derivation of smoothing distributions and the calculation of likelihood functions. See [Cappé, Moulines and Rydén \(2005\)](#) for a book-length treatment of the recursions involved in such computations and their dependence on the filtering distributions.

The literature on HMMs has largely focussed on parametric signals, where the unobserved Markov process is finite dimensional. Recently, authors have considered infinite dimensional HMMs, which involve the dynamics of infinitely many parameters. One strand of work originates with [Beal, Ghahramani and Rasmussen \(2002\)](#), who model the signal as a Markov chain with countable state space and transitions based on a hierarchy of Dirichlet processes. See also [Van Gael, Saatchi, Teh and Ghahramani \(2008\)](#), [Stepleton, Ghahramani, Gordon and Lee \(2009\)](#) and [Zhang, Zhu and Zhang \(2014\)](#) for further developments. A different strand of work tries to build time-evolving Dirichlet processes for semi-parametric time-series analysis; see for example [Griffin and Steel \(2006\)](#), [Rodriguez and ter Horst \(2008\)](#) and [Mena and Ruggiero \(2014\)](#), which all build upon the celebrated stick breaking representation of the Dirichlet process ([Sethuraman, 1994](#)). Finally, yet another class of infinite dimensional HMMs takes the signal as a Markov chain with finite state space but uses an infinite number of parameters for the emission distribution, see [Yau, Papaspiliopoulos, Roberts and Holmes \(2011\)](#). A common feature of the above mentioned contributions is that they all resort to Monte Carlo strategies for posterior computation.

In this paper we study models for time-evolving measures and derive their posterior distributions analytically. We consider two families of models that give rise to infinite dimensional or measure-valued hidden Markov models, and term these families *hidden Markov measures*. In the first family we consider two models. The simpler of the two assumes that the signal at each time point is a probability distribution on a countable set, with infinitely many parameters. The signal evolves in continuous time according an infinite Wright–Fisher diffusion, and the data are available in discrete times and are realisations from the distribution given by the signal. In the more general model, the signal at each time point is a discrete distribution on a Polish space, evolving as a Fleming–Viot diffusion, and the data are obtained in discrete times as draws from the underlying measure. The former model admits a likelihood and can be dealt with by means of direct methods, whereas the latter features an evolving support for the signal states and needs to be manipulated indirectly, hence they are treated separately. In the second family the signal at each time point is a positive almost surely discrete measure on a Polish space, the signal evolves in continuous time according to the Dawson–Watanabe diffusion, and

the data are available in discrete times as realisations from a doubly stochastic Poisson process with intensity given by the signal.

In our specification, the Fleming–Viot and Dawson–Watanabe processes are stationary with respect to the laws of Dirichlet and gamma random measures respectively. Additionally, the processes can be defined so that one parameter controls the correlation structure and other parameters determine the invariant distribution. Therefore, our models are natural *dynamic* extensions of infinite dimensional *static* models for unknown distributions and intensities that are widely used in Bayesian statistics and machine learning for a broad range of applications. A fundamental reason for the popularity of the static models are *conjugacy* properties that make the Bayesian updating tractable.

In this paper we demonstrate that Bayesian learning is tractable for the families of hidden Markov measures we consider. We show that the filtering distributions evolve within finite mixtures of Dirichlet and gamma random measures, and we provide a recursive algorithm for the computation of the parameters of these mixtures. Broadly speaking, our theory builds upon a synthesis of three classes of background results. The first is the connection between filtering and the so-called dual process, which was recently established in [Papaspiliopoulos and Ruggiero \(2014\)](#) and is reviewed in Section 2. This previous work identifies classes of parametric HMMs for which the filtering distributions evolve in finite mixtures of finite dimensional distributions and provides a recursive algorithm for the associated parameters. The second class of results is concerned with the projective properties of the Fleming–Viot and Dawson–Watanabe processes that link them to the finite dimensional Wright–Fisher and Cox–Ingersoll–Ross processes; the details are discussed in Sections 3.2 and 4.2 respectively. The third relates to the conjugacy properties of the corresponding static models and mixtures thereof, discussed in Sections 3.1 and 4.1. Our strategy exploits the fact that the finite dimensional projected models admit computable filters, due to the results in [Papaspiliopoulos and Ruggiero \(2014\)](#), and that the exchange of the operations “projection” and “filter” is valid, which we prove in this paper. Figure 1 depicts the strategy for obtaining our results. Given the signal distribution $\mathcal{L}(X_{t_n} \mid Y_{1:n})$, its time propagation $\mathcal{L}(X_{t_{n+k}} \mid Y_{1:n})$ is found by propagating in time the projection of the former onto an arbitrary partition, that is $\mathcal{L}(X_{t_n}(A_1, \dots, A_K) \mid Y_{1:n})$, and by exploiting the projective characterisation of the filtering distributions. This is done in Theorems 3.3 and 4.3 for the Fleming–Viot and Dawson–Watanabe case respectively, while the same result is obtained in Theorem 3.1 for the infinite alleles model by exploiting directly the duality properties. In addition, a duality result and the time propagation are provided in Theorem 4.1 and Proposition 4.2 for a class of multivariate Cox–Ingersoll–Ross processes.

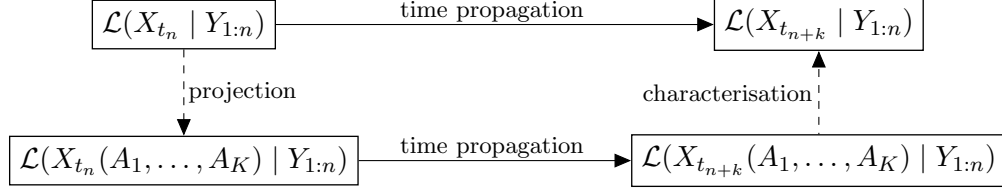


Figure 1: Given the same amount of data, the time propagation of the filtering distribution $\mathcal{L}(X_{t_n} | Y_{1:n})$ is found by propagating its projection onto an arbitrary partition (A_1, \dots, A_K) , and by exploiting the projective characterisation of the filtering distributions.

1.2 Notations

Throughout the paper, \mathcal{Y} will denote a locally compact Polish space, $\mathcal{M}(\mathcal{Y})$ the space of finite Borel measures on \mathcal{Y} , $\mathcal{M}_1(\mathcal{Y})$ its subspace of probability measures. A typical element of $\mathcal{M}(\mathcal{Y})$ will be

$$(1) \quad \alpha \in \mathcal{M}(\mathcal{Y}), \quad \alpha = \theta P_0, \quad \theta > 0, \quad P_0 \in \mathcal{M}_1(\mathcal{Y}),$$

where θ is the total mass of α . The discrete measures $x(\cdot) \in \mathcal{M}_1(\mathcal{Y})$ and $z(\cdot) \in \mathcal{M}(\mathcal{Y})$ will denote the marginal states of the signals X_t and Z_t , with $X_t(A)$ and $Z_t(A)$ being the respective one dimensional projections onto the Borel set $A \subset \mathcal{Y}$. We will also adopt boldface notation to denote vectors, with the following conventions:

$$\mathbf{x} = (x_1, \dots, x_K) \in \mathbb{R}_+^K, \quad \mathbf{m} = (m_1, \dots, m_K) \in \mathbb{Z}_+^K, \quad \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_K^{m_K}, \quad |\mathbf{x}| = \sum_{i=1}^K x_i,$$

where the dimension $2 \leq K \leq \infty$ will be clear from the context if not specified explicitly. Typically, \mathbf{x} will represent a finite dimensional signal state and \mathbf{m} a vector of multiplicities.

2 Computable filtering and duality

Computable filtering refers to the circumstance where the filtering distributions can be characterised by a finite number of parameters whose computation can be achieved at cost that grows at most polynomially with the number of observations. Special cases of this framework are finite dimensional filters for which the computational cost is linear in the number of observations, the Kalman filter for linear Gaussian HMMs being the celebrated model in this setting.

Papaspiliopoulos and Ruggiero (2014) recently developed a framework for the computable filtering of finite dimensional HMMs. A brief description of the framework is as follows. There is a finite dimensional HMM $\{(X_{t_n}, Y_{t_n}), n \geq 0\}$, where X_t has state space \mathcal{X} , stationary distribution π , transition kernel $P_t(x, dx')$, and the data are linked to the signal via an emission density $f_x(y)$. The filtering distributions are $\nu_n := \mathcal{L}(X_{t_n} | Y_{1:n})$ and ν is the prior distribution for X_{t_0} . The exact or optimal filter is the solution of the recursion

$$\nu_0 = \phi_{Y_{t_0}}(\nu), \quad \nu_n = \phi_{Y_{t_n}}(\psi_{t_n - t_{n-1}}(\nu_{n-1})), \quad n \in \mathbb{N},$$

which involves the following two operators acting on measures: the *update operator*

$$(2) \quad \phi_y(\nu)(dx) = \frac{f_x(y)\nu(dx)}{p_\nu(y)}, \quad p_\nu(y) = \int_{\mathcal{X}} f_x(y)\nu(dx),$$

and the *prediction operator*

$$(3) \quad \psi_t(\nu)(dx') = \int_{\mathcal{X}} \nu(dx) P_t(x, dx').$$

The existence of a computable filter and a recursive algorithm can be established if the following structure is embedded in the HMM:

- *Conjugacy*: there exists a function $h(x, \mathbf{m}, \theta) \geq 0$, where $x \in \mathcal{X}$, $\mathbf{m} \in \mathbb{Z}_+^K$ for some $K \in \mathbb{N}$, and $\theta \in \mathbb{R}^l$ for some $l \in \mathbb{N}$, and functions $t(y, \mathbf{m})$ and $T(y, \theta)$ such that $\int h(x, \mathbf{m}, \theta)\pi(dx) = 1$, for all \mathbf{m} and θ , and

$$\phi_y(h(x, \mathbf{m}, \theta)\pi(dx)) = h(x, t(y, \mathbf{m}), T(y, \theta))\pi(dx).$$

- *Duality*: there exists a two-component Markov process (M_t, Θ_t) with state-space $\mathbb{Z}_+^K \times \mathbb{R}^l$, and generator,

$$(Ag)(\mathbf{m}, \theta) = \lambda(|\mathbf{m}|)\rho(\theta) \sum_{i=1}^K m_i [g(\mathbf{m} - \mathbf{e}_i, \theta) - g(\mathbf{m}, \theta)] + \sum_{i=1}^l r_i(\theta) \frac{\partial g(\mathbf{m}, \theta)}{\partial \theta_i},$$

such that it is *dual* to X_t with respect to the function h , i.e., it satisfies

$$(4) \quad \mathbb{E}^x[h(X_t, \mathbf{m}, \theta)] = \mathbb{E}^{(\mathbf{m}, \theta)}[h(x, M_t, \Theta_t)], \quad \forall x \in \mathcal{X}, \mathbf{m} \in \mathbb{Z}_+^K, \theta \in \mathbb{R}^l, t \geq 0.$$

Note that the Θ component of the dual process is assumed to evolve autonomously, according to a system of ordinary differential equations, and modulates the rates of M_t , which is a death process on a K -dimensional lattice. Under these conditions, Proposition 2.3 of Papaspiliopoulos and Ruggiero

(2014) shows that for $\mathcal{F} = \{h(x, \mathbf{m}, \theta)\pi(dx), \mathbf{m} \in \mathbb{Z}_+^K, \theta \in \mathbb{R}^l\}$, if $\nu \in \mathcal{F}$, then ν_n is a finite mixture of distributions in \mathcal{F} with parameters that can be computed recursively. Additionally, the local sufficient condition for duality

$$(5) \quad (\mathcal{A}h(\cdot, \mathbf{m}, \theta))(x) = (\mathcal{A}h(x, \cdot, \cdot))(\mathbf{m}, \theta), \quad \forall x \in \mathcal{X}, \mathbf{m} \in \mathbb{Z}_+^K, \theta \in \mathbb{R}^l,$$

where \mathcal{A} denotes the generator of X_t , is applied to identify dual processes and computable filters when the signal is the Cox–Ingersoll–Ross or the K -dimensional Wright–Fisher diffusion. The work of Papaspiliopoulos and Ruggiero (2014) includes as special cases computable filters obtained previously in Genon-Catalot and Kessler (2004) and Chaleyat-Maurel and Genon-Catalot (2006; 2009). However, it is strictly applicable to finite dimensional signals, due to the assumptions that an emission density and a finite dimensional dual exist.

3 Filtering Fleming–Viot processes

3.1 The static model: Dirichlet process

The Dirichlet process, introduced by Ferguson (1973) and commonly recognised as the cornerstone in Bayesian nonparametrics (see Ghosal (2010) for a recent review), is a discrete random probability measure $x \in \mathcal{M}_1(\mathcal{Y})$ that can be thought to describe the frequencies in a population with infinitely many labelled types, whereby \mathcal{Y} is often referred to as the *type space*. The process admits the series representation

$$(6) \quad x(\cdot) = \sum_{i=1}^{\infty} W_i \delta_{Y_i}(\cdot), \quad W_i = \frac{Q_i}{\sum_{j \geq 1} Q_j}, \quad Y_i \stackrel{iid}{\sim} P_0, \quad (Y_i) \perp (W_i).$$

Here $\{Q_i, i \geq 1\}$ are the jumps of a gamma process with mean measure $\theta y^{-1} e^{-y} dy$. We will denote by Π_α , $\alpha = \theta P_0$, the law of $x(\cdot)$ in (6). The Dirichlet process has two fundamental properties that are of great interest in statistical learning:

- *Conjugacy*: $y_i \mid x \stackrel{iid}{\sim} x$ and $x \sim \Pi_\alpha$ imply $x \mid \mathbf{y}_{1:m} \sim \Pi_{\alpha + \sum_{i=1}^m \delta_{y_i}}$, where $\mathbf{y}_{1:m} := (y_1, \dots, y_m)$.
- *Projection*: for a measurable partition A_1, \dots, A_K of \mathcal{Y} , the vector $(x(A_1), \dots, x(A_K))$ has the Dirichlet distribution with parameter $\boldsymbol{\alpha} = (\alpha(A_1), \dots, \alpha(A_K))$, henceforth denoted $\pi_{\boldsymbol{\alpha}}$.

When $\mathcal{Y} = \mathbb{N}$, the Dirichlet process can be seen as a Dirichlet distribution with infinitely many types defined on

$$(7) \quad \Delta_\infty = \left\{ x \in [0, 1]^\infty : \sum_{i=1}^{\infty} x_i = 1 \right\}.$$

This corresponds to the construction

$$(8) \quad V_i = \frac{T_i}{\sum_{j \geq 1} T_j}, \quad T_j \stackrel{ind}{\sim} \text{Ga}(\alpha_j, 1),$$

where

$$(9) \quad \alpha = (\alpha_1, \alpha_2, \dots), \quad \alpha_i = \theta P_0(\{i\}), \quad i \geq 1.$$

See [Ethier and Griffiths \(1993\)](#), equation (1.26) and Lemma 2.2. With a slight abuse of notation, we will denote the law of (V_1, V_2, \dots) by the same symbol π_α used for the Dirichlet distribution; which distribution π_α refers to will be clear from the context.

Mixtures of Dirichlet processes were introduced in [Antoniak \(1974\)](#). They add a further level to the Bayesian hierarchical model, whereby

$$y_i \mid x, u \stackrel{iid}{\sim} x, \quad x \mid u \sim \Pi_{\alpha_u}, \quad u \sim H,$$

where α_u denotes the measure α conditionally on u . Equivalently,

$$y_i \mid x \stackrel{iid}{\sim} x, \quad x \sim \int_{\mathcal{U}} \Pi_{\alpha_u} dH(u).$$

For mixtures of Dirichlet processes the conjugacy and projective properties read as follows:

- *Conjugacy*: $y_i \mid x \stackrel{iid}{\sim} x$ and $x \sim \int_{\mathcal{U}} \Pi_{\alpha_u} dH(u)$ imply

$$x \mid \mathbf{y}_{1:m} \sim \int_{\mathcal{U}} \Pi_{\alpha_u + \sum_{i=1}^m \delta_{y_i}} dH_{\mathbf{y}_{1:m}}(u),$$

where $H_{\mathbf{y}_{1:m}}$ is the conditional distribution of u given $\mathbf{y}_{1:m}$.

- *Projection*: for a measurable partition A_1, \dots, A_K of \mathcal{Y} , we have

$$(x(A_1), \dots, x(A_K)) \sim \int_{\mathcal{U}} \pi_{\alpha_u} dH(u),$$

where $\alpha_u = (\alpha_u(A_1), \dots, \alpha_u(A_K))$.

3.2 The signal: Fleming–Viot process

Fleming–Viot (FV) processes are a family of diffusions taking values in the subspace of $\mathcal{M}_1(\mathcal{Y})$ given by purely atomic probability measures, hence they describe evolving discrete distributions. See [Ethier and Kurtz \(1993\)](#) and [Dawson \(1993\)](#) for exhaustive reviews. Here we restrict the attention to a subclass known as the (labelled) *infinitely many neutral alleles model* with parent independent mutation, henceforth for simplicity called the FV process. This is characterised by the generator

$$(10) \quad \mathbb{A}\varphi(x) = \frac{1}{2} \int_{\mathcal{Y}} \int_{\mathcal{Y}} x(dy)(\delta_y(du) - x(du)) \frac{\partial^2 \varphi(x)}{\partial x(dy) \partial x(du)} + \int_{\mathcal{Y}} x(dy) B\left(\frac{\partial \varphi(x)}{\partial x(\cdot)}\right)(y)$$

where φ is a test function, δ_y is a point mass at y and $\partial \varphi(x)/\partial x(dy) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1}(\varphi(x + \varepsilon \delta_y) - \varphi(x))$. Furthermore, B is the *mutation operator*, that is the generator of a Feller pure-jump process on \mathcal{Y}

$$(11) \quad (Bf)(y) = \frac{\theta}{2} \int_{\mathcal{Y}} [f(u) - f(y)] P_0(du), \quad f \in C(\mathcal{Y}),$$

whereby at rate $\theta/2$ jumps occur to a location sampled from $P_0 \in \mathcal{M}_1(\mathcal{Y})$ independently of the current value of the jump process, whence mutations are *parent independent*. The domain of \mathbb{A} is taken to be the set of $\varphi \in C(\mathcal{M}_1(\mathcal{Y}))$ of the form $\varphi(x) = F(\langle f_1, x \rangle, \dots, \langle f_k, x \rangle)$, where $f_i \in C(\mathcal{Y})$ and $F \in C^2(\mathbb{R}^k)$. This FV process is known to be stationary and reversible with respect to the law Π_α of a Dirichlet process as in [\(6\)](#); see [Ethier and Kurtz \(1993\)](#), Section 8.

Projecting a FV process X_t onto a measurable partition A_1, \dots, A_K of \mathcal{Y} yields a K -dimensional Wright–Fisher (WF) diffusion, which is reversible and stationary with respect to the Dirichlet distribution π_α , for $\alpha_i = \theta P_0(A_i)$, $i = 1, \dots, K$, and has infinitesimal operator, for $x_i = x(A_i)$,

$$(12) \quad \mathcal{A}_K f(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^K x_i(\delta_{ij} - x_j) \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} + \frac{1}{2} \sum_{i=1}^K (\alpha_i - \theta x_i) \frac{\partial f(\mathbf{x})}{\partial x_i}.$$

Here δ_{ij} denotes Kronecker delta and \mathcal{A}_K acts on $C^2(\Delta_K)$ functions. See [Dawson \(2010\)](#). This property is the dynamic counterpart of the projective property of Dirichlet processes discussed earlier. The same result is obtained when the mutant type distribution P_0 is finitely supported, for example when $\mathcal{Y} = \{1, \dots, K\}$.

When K goes to infinity in [\(12\)](#), or alternatively when $\mathcal{Y} = \mathbb{N}$ in [\(10\)](#), the signal follows a diffusion characterised in [Ethier \(1981\)](#), with generator

$$(13) \quad \mathcal{A}_\infty f(\mathbf{x}) = \frac{1}{2} \sum_{i,j=1}^\infty x_i(\delta_{ij} - x_j) \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} - \frac{1}{2} \sum_{i=1}^\infty (\alpha_i - \theta x_i) \frac{\partial f(\mathbf{x})}{\partial x_i},$$

where $x_i = X(\{i\})$ and $(\alpha_1, \alpha_2, \dots)$ is as in (9). Here \mathcal{A}_∞ acts on functions $C^2(\overline{\Delta}_\infty)$ which depend on finitely many coordinates, $\overline{\Delta}_\infty$ denoting the closure of (7) in the product topology, and the associated process is reversible with respect to the law π_α of (8). See also [Ethier and Griffiths \(1993\)](#) and [Ethier and Kurtz \(1993\)](#). In this paper we will refer to this process as the *infinite WF diffusion*, given its structural similarity with (12).

For statistical modelling it is useful to introduce a further parameter σ that controls the speed of the process. This can be done by defining $\mathbb{A}' = \sigma\mathbb{A}$, $\mathcal{A}'_K = \sigma\mathcal{A}_K$ and $\mathcal{A}'_\infty = \sigma\mathcal{A}_\infty$, which correspond to the time change $X_{\tau(t)}$ with $\tau(t) = \sigma t$. In such parameterisation, σ does not affect the stationary distribution of the process, and can be used to model the dependence structure. For simplicity of exposition, the theory below focuses on the case $\sigma = 1$.

3.3 Filtering infinite Wright–Fisher diffusions

We consider a model for evolving distributions with countable support. In particular, suppose the signal \mathbf{X}_t follows an infinite WF diffusion with generator (13), and assume that, given $\mathbf{X}_t = \mathbf{x} \in \Delta_\infty$, the emission distribution is multinomial with density

$$(14) \quad \text{MN}(\mathbf{m}; |\mathbf{m}|, \mathbf{x}) = \binom{|\mathbf{m}|}{\mathbf{m}} \prod_{j \geq 1} x_j^{m_j}, \quad |\mathbf{m}| < \infty.$$

Note that the above probability mass function is well defined for $\mathbf{m} \in \mathbb{Z}_+^\infty$ such that $|\mathbf{m}| < \infty$. The following Theorem provides the prediction step of the filtering algorithm, extending a result obtained in [Papaspiliopoulos and Ruggiero \(2014\)](#) for finite K .

Theorem 3.1. *Let \mathbf{X}_t have generator (13) and π_α be the law of (8). Then, for any $\mathbf{m} \in \mathbb{Z}_+^\infty$ such that $|\mathbf{m}| < \infty$,*

$$(15) \quad \psi_t(\pi_{\alpha+\mathbf{m}}) = \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{m}} p_{\mathbf{m}, \mathbf{m}-\mathbf{i}}(t) \pi_{\alpha+\mathbf{m}-\mathbf{i}},$$

where

$$(16) \quad \begin{aligned} p_{\mathbf{m}, \mathbf{m}}(t) &= e^{-\lambda_{|\mathbf{m}|} t}, \\ p_{\mathbf{m}, \mathbf{m}-\mathbf{i}}(t) &= C_{|\mathbf{m}|, |\mathbf{m}|-|\mathbf{i}|}(t) \left(\prod_{h=0}^{|\mathbf{i}|-1} \lambda_{|\mathbf{m}|-h} \right) p(\mathbf{i}; \mathbf{m}, |\mathbf{i}|), \quad \mathbf{0} < \mathbf{i} \leq \mathbf{m}, \\ C_{|\mathbf{m}|, |\mathbf{m}|-|\mathbf{i}|}(t) &= (-1)^{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|} \frac{e^{-\lambda_{|\mathbf{m}|-k} t}}{\prod_{0 \leq h \leq |\mathbf{i}|, h \neq k} (\lambda_{|\mathbf{m}|-k} - \lambda_{|\mathbf{m}|-h})}, \end{aligned}$$

and where

$$(17) \quad p(\mathbf{i}; \mathbf{m}, |\mathbf{i}|) = \binom{|\mathbf{m}|}{|\mathbf{i}|}^{-1} \prod_{j \geq 1} \binom{m_j}{i_j}$$

is the multivariate hypergeometric probability function, with parameters $(\mathbf{m}, |\mathbf{i}|)$, evaluated at \mathbf{i} .

Proof. Define

$$(18) \quad h(\mathbf{x}, \mathbf{m}) = \frac{\Gamma(\theta + |\mathbf{m}|)}{\Gamma(\theta)} \prod_{j \geq 1} \frac{\Gamma(\alpha_j)}{\Gamma(\alpha_j + m_j)} \mathbf{x}^{\mathbf{m}}, \quad \mathbf{m} \in \mathbb{Z}_+^\infty, |\mathbf{m}| < \infty,$$

which is in the domain of \mathcal{A}_∞ , with $(\alpha_1, \alpha_2, \dots)$ as in (9) and $m_j = 0$ if $\alpha_j = 0$. Let also \mathbf{e}_i be the vector whose only non zero component is a 1 at the i th coordinate. A direct computation shows that

$$\begin{aligned} \mathcal{A}_\infty h(\mathbf{x}, \mathbf{m}) &= \sum_{i \geq 1} \left(\frac{\alpha_i m_i}{2} + \binom{m_i}{2} \right) \frac{\Gamma(\theta + |\mathbf{m}|)}{\Gamma(\theta)} \prod_{j \geq 1} \frac{\Gamma(\alpha_j)}{\Gamma(\alpha_j + m_j)} \mathbf{x}^{\mathbf{m} - \mathbf{e}_i} \\ &\quad - \sum_{i \geq 1} \left(\frac{\theta m_i}{2} + \binom{m_i}{2} + \frac{1}{2} m_i \sum_{j \neq i} m_j \right) \frac{\Gamma(\theta + |\mathbf{m}|)}{\Gamma(\theta)} \prod_{j \geq 1} \frac{\Gamma(\alpha_j)}{\Gamma(\alpha_j + m_j)} \mathbf{x}^{\mathbf{m}} \\ &= \frac{\theta + |\mathbf{m}| - 1}{2} \sum_{i \geq 1} m_i h(\mathbf{x}, \mathbf{m} - \mathbf{e}_i) - \frac{|\mathbf{m}|(\theta + |\mathbf{m}| - 1)}{2} h(\mathbf{x}, \mathbf{m}). \end{aligned}$$

Hence, by (5), the death process M_t on \mathbb{Z}_+^∞ , which jumps from \mathbf{m} to $\mathbf{m} - \mathbf{e}_i$ at rate $m_i(\theta + |\mathbf{m}| - 1)/2$, is dual to the infinite Dirichlet diffusion with generator \mathcal{A}_∞ with respect to (18). From the definition (3) of the prediction operator now we have

$$\begin{aligned} \psi_t(\pi_{\alpha+\mathbf{m}})(d\mathbf{x}') &= \int_{\mathcal{X}} h(\mathbf{x}, \mathbf{m}) \pi_\alpha(d\mathbf{x}) P_t(\mathbf{x}, d\mathbf{x}') \\ &= \int_{\mathcal{X}} h(\mathbf{x}, \mathbf{m}) \pi_\alpha(d\mathbf{x}') P_t(\mathbf{x}', d\mathbf{x}) \\ &= \pi_\alpha(d\mathbf{x}') \mathbb{E}^{\mathbf{x}'}[h(\mathbf{X}_t, \mathbf{m})] \\ &= \pi_\alpha(d\mathbf{x}') \mathbb{E}^{\mathbf{m}}[h(\mathbf{x}', M_t)] \\ &= \pi_\alpha(d\mathbf{x}') \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{m}} p_{\mathbf{m}, \mathbf{m}-\mathbf{i}}(t) h(\mathbf{x}', \mathbf{m} - \mathbf{i}) \\ &= \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{m}} p_{\mathbf{m}, \mathbf{m}-\mathbf{i}}(t) \pi_{\alpha+\mathbf{m}-\mathbf{i}}(d\mathbf{x}') \end{aligned}$$

where the second equality holds in virtue of the reversibility of \mathbf{X}_t with respect to π_α , the fourth by the duality established above together with (4) and the fifth from Lemma A.1 in the Appendix. \square

Assuming now an observation $\mathbf{y} = (y_1, y_2, \dots)$ has the extended multinomial likelihood (14), the update step (2) in this case becomes

$$(19) \quad \phi_{\mathbf{y}}(\pi_{\alpha+\mathbf{m}}) = \pi_{\alpha+\mathbf{m}+\mathbf{y}},$$

which follows from the conjugacy of the Dirichlet process (see Section 3.1) by taking $\mathcal{Y} = \mathbb{N}$.

The following Proposition summarises the findings of this section by providing the learning algorithm that allows computable filtering in this framework. Let

$$(20) \quad \mathcal{M} = \{\mathbf{m} = (m_1, \dots, m_K) \in \mathbb{Z}_+^K, K \in \mathbb{N}\},$$

with a partial ordering defined by “ $<$ ”, so that $\mathbf{m} < \mathbf{n}$ if $m_j \leq n_j$ for all $j \geq 1$ and $m_j < n_j$ for some $j \geq 1$. For $M \subset \mathcal{M}$, let also

$$(21) \quad G(M) = \{\mathbf{n} \in \mathcal{M} : \mathbf{0} \leq \mathbf{n} \leq \mathbf{m}, \mathbf{m} \in M\}$$

be the set of nonnegative vectors lying beneath those in M .

Proposition 3.2. *Consider the family of finite mixtures*

$$\mathcal{F} = \left\{ \sum_{\mathbf{m} \in \Lambda} w_{\mathbf{m}} h(\mathbf{x}, \mathbf{m}) \pi_{\alpha}(\mathrm{d}\mathbf{x}), \Lambda \subset \mathcal{M}, |\Lambda| < \infty, w_{\mathbf{m}} \geq 0, \sum_{\mathbf{m} \in \Lambda} w_{\mathbf{m}} = 1 \right\},$$

where π_{α} is the law of (8) and $h(\mathbf{x}, \mathbf{m})$ is as in (18). Then, when \mathbf{X}_t has generator (13) and data are as in (14), \mathcal{F} is closed under the application of the update and prediction operators (2) and (3). Specifically,

$$(22) \quad \phi_{\mathbf{y}} \left(\sum_{\mathbf{m} \in \Lambda} w_{\mathbf{m}} h(\mathbf{x}, \mathbf{m}) \pi_{\alpha}(\mathrm{d}\mathbf{x}) \right) = \sum_{\mathbf{n} \in t(\mathbf{y}, \Lambda)} \hat{w}_{\mathbf{n}} h(\mathbf{x}, \mathbf{n}) \pi_{\alpha}(\mathrm{d}\mathbf{x}),$$

with

$$(23) \quad t(\mathbf{y}, \Lambda) := \{\mathbf{n} : \mathbf{n} = t(\mathbf{y}, \mathbf{m}), \mathbf{m} \in \Lambda\}$$

$$\hat{w}_{\mathbf{n}} \propto w_{\mathbf{m}} \frac{\Gamma(\theta + |\mathbf{m}|)}{\Gamma(\theta + |\mathbf{m}| + |\mathbf{y}|)} \prod_{j \geq 1} \frac{\Gamma(\alpha_j + \mathbf{m}_j + y_j)}{\Gamma(\alpha_j + \mathbf{m}_j)} \quad \text{for } \mathbf{n} = t(\mathbf{y}, \mathbf{m}), \sum_{\mathbf{n} \in t(\mathbf{y}, \Lambda)} \hat{w}_{\mathbf{n}} = 1,$$

and

$$(24) \quad \psi_t \left(\sum_{\mathbf{m} \in \Lambda} w_{\mathbf{m}} h(\mathbf{x}, \mathbf{m}) \pi_{\alpha}(\mathrm{d}\mathbf{x}) \right) = \sum_{\mathbf{n} \in G(\Lambda)} \left(\sum_{\mathbf{m} \in \Lambda, \mathbf{m} \geq \mathbf{n}} w_{\mathbf{m}} p_{\mathbf{m}, \mathbf{n}}(t) \right) h(\mathbf{x}, \mathbf{n}) \pi_{\alpha}(\mathrm{d}\mathbf{x}).$$

Proof. The update operation, given by (22) and (23), follows from (19) together with the fact that for a mixture $\sum_{i=1}^n w_i \nu_i$ we have

$$(25) \quad \phi_{\mathbf{y}} \left(\sum_{i=1}^n w_i \nu_i \right) = \sum_{i=1}^n \frac{w_i p_{\nu_i}(\mathbf{y})}{\sum_j w_j p_{\nu_j}(\mathbf{y})} \phi_{\mathbf{y}}(\nu_i), \quad p_{\nu_i}(\mathbf{y}) = \int_{\mathcal{X}} f_{\mathbf{x}}(\mathbf{y}) \nu_i(d\mathbf{x}),$$

while (24) follows from the fact that

$$(26) \quad \psi_t \left(\sum_{i=1}^n w_i \nu_i \right) = \sum_{i=1}^n w_i \psi_t(\nu_i)$$

together with Theorem 3.1 and a rearrangement of the sums. \square

3.4 Filtering Fleming–Viot processes

Let now the signal X_t be a FV process with generator (10). We assume that given the signal state $X_t = x \in \mathcal{M}_1$, observations are drawn independently from x , i.e.,

$$(27) \quad y_i \mid x \stackrel{iid}{\sim} x.$$

A sample $\mathbf{y}_{1:m} = (y_1, \dots, y_m)$ from the discrete distribution x will feature ties among the observations with positive probability. Denote by $(y_1^*, \dots, y_{K_m}^*)$ the distinct values in $\mathbf{y}_{1:m}$ and by $\mathbf{m} = (m_1, \dots, m_{K_m})$ the associated multiplicities, so that $|\mathbf{m}| = m$. The following result provides the prediction step for filtering FV processes.

Theorem 3.3. *Let X_t have generator (10), with invariant law Π_{α} . Then, for any $\mathbf{y}_{1:m}$ with multiplicities \mathbf{m} ,*

$$(28) \quad \psi_t \left(\Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}} \right) = \sum_{\mathbf{n} \in G(\mathbf{m})} p_{\mathbf{m}, \mathbf{n}}(t) \Pi_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}},$$

with $p_{\mathbf{m}, \mathbf{n}}(t)$ as in (16) and G as in (21).

Proof. Fix an arbitrary partition (A_1, \dots, A_K) of \mathcal{Y} with K classes, and denote by $\tilde{\mathbf{m}}$ the multiplicities resulting from binning $\mathbf{y}_{1:m}$ into the corresponding cells. Then

$$(29) \quad \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}(A_1, \dots, A_K) \sim \pi_{\alpha + \tilde{\mathbf{m}}},$$

where $\Pi_{\alpha+\sum_{i=1}^{K_m} m_i \delta_{y_i^*}}(A_1, \dots, A_K)$ denotes $\Pi_{\alpha+\sum_{i=1}^{K_m} m_i \delta_{y_i^*}}(\cdot)$ evaluated on (A_1, \dots, A_K) . Since the projection onto the same partition of the FV process is a K -dimensional WF process (see Section 3.2), from Theorem 3.1 we have

$$(30) \quad \begin{aligned} \psi_t \left(\Pi_{\alpha+\sum_{i=1}^{K_m} m_i \delta_{y_i^*}}(A_1, \dots, A_K) \right) &= \psi_t(\pi_{\alpha+\tilde{\mathbf{m}}}) \\ &= \sum_{\mathbf{n} \in G(\tilde{\mathbf{m}})} p_{\tilde{\mathbf{m}}, \mathbf{n}}(t) \pi_{\alpha+\mathbf{n}}. \end{aligned}$$

Furthermore, since a Dirichlet process is characterised by its finite-dimensional projections, now it suffices to show that

$$\sum_{\mathbf{n} \in G(\mathbf{m})} p_{\mathbf{m}, \mathbf{n}}(t) \Pi_{\alpha+\sum_{i=1}^{K_m} n_i \delta_{y_i^*}}(A_1, \dots, A_K) = \sum_{\mathbf{n} \in G(\tilde{\mathbf{m}})} p_{\tilde{\mathbf{m}}, \mathbf{n}}(t) \pi_{\alpha+\mathbf{n}}$$

so that the operations of propagation and projection commute. Given (29), we only need to show that the mixture weights are consistent with respect to fragmentation and merging of classes, that is

$$\sum_{\mathbf{i} \in G(\mathbf{m}): \tilde{\mathbf{i}}=\mathbf{n}} p_{\mathbf{m}, \mathbf{i}}(t) = p_{\tilde{\mathbf{m}}, \mathbf{n}}(t),$$

where $\tilde{\mathbf{i}}$ denotes the projection of \mathbf{i} onto (A_1, \dots, A_K) . Using (16), the previous in turn reduces to

$$(31) \quad \sum_{\mathbf{i} \in G(\mathbf{m}): \tilde{\mathbf{i}}=\mathbf{n}} p(\mathbf{i}; \mathbf{m}, m-i) = p(\mathbf{n}; \tilde{\mathbf{m}}, m-n),$$

which holds by the marginalization properties of the multivariate hypergeometric distribution. Cf. Johnson, Kotz and Balakrishnan (1997), equation 39.3. \square

Hence a single Dirichlet measure evolves into a finite mixture of Dirichlet measures. Note that when $\mathbf{m} = (0, \dots, 0)$, (28) reduces to the stationarity equation for FV processes.

We now turn to the update step. Let $(y_{K_m+1}^*, \dots, y_{K_m+n}^*)$ be the distinct values observed in an additional sample $\mathbf{y}_{m+1:m+n}$ of size n that are not included in $(y_1^*, \dots, y_{K_m}^*)$, and let \mathbf{n} be the multiplicities of the full vector of distinct values $(y_1^*, \dots, y_{K_m+n}^*)$. Denote also by $\text{PU}_\alpha(\mathbf{y}_{m+1:m+n} \mid \mathbf{y}_{1:m})$ the joint distribution of $\mathbf{y}_{m+1:m+n}$ sampled from a conditional Blackwell–MacQueen Pólya urn scheme (Blackwell and MacQueen, 1973), i.e.,

$$Y_{m+i+1} \mid \mathbf{y}_{1:m+i} \sim \frac{\theta P_0 + \sum_{j=1}^{m+i} \delta_{y_j}}{\theta + m + i}, \quad i = 0, \dots, n-1.$$

The following result, stated here in our notation for ease of the reader, is a special case of [Antoniak \(1974\)](#).

Lemma 3.4. *Let $y_{m+i} \mid x \sim x$, $i = 1, \dots, n$, with*

$$x \sim \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}, \quad \sum_{\mathbf{m} \in M} w_{\mathbf{m}} = 1.$$

Then

$$(32) \quad \phi_{\mathbf{y}_{m+1:m+n}} \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}} \right) = \sum_{\mathbf{n} \in t(\mathbf{y}_{m+1:m+n}, M)} \hat{w}_{\mathbf{n}} \Pi_{\alpha + \sum_{i=1}^{K_{m+n}} n_i \delta_{y_i^*}},$$

where $t(\cdot)$ is as in [Proposition 3.2](#) and

$$(33) \quad \hat{w}_{\mathbf{n}} \propto w_{\mathbf{m}} \text{PU}_{\alpha}(\mathbf{y}_{m+1:m+n} \mid \mathbf{y}_{1:m}).$$

Proof. The distribution x is a mixture of Dirichlet processes with mixing measure $H(\cdot) = \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \delta_{\mathbf{m}}(\cdot)$ on M and transition measure

$$\alpha_{\mathbf{m}}(\cdot) = \alpha(\cdot) + \sum_{j=1}^{K_m} m_j \delta_{y_j^*}(\cdot) = \alpha(\cdot) + \sum_{i=1}^m \delta_{y_i}(\cdot),$$

where $\mathbf{y}_{1:m}$ is the full sample. See Section 3.1. Lemma 1 and Corollary 3.2' in [Antoniak \(1974\)](#) now imply that

$$x \mid \mathbf{m}, \mathbf{y}_{m+1:m+n} \sim \Pi_{\alpha_{\mathbf{m}}(\cdot) + \sum_{i=m+1}^n \delta_{y_i}(\cdot)} = \Pi_{\alpha(\cdot) + \sum_{i=1}^n \delta_{y_i}}$$

and $H(\mathbf{m} \mid \mathbf{y}_{m+1:m+n}) \propto w_{\mathbf{m}} \text{PU}_{\alpha}(\mathbf{y}_{m+1:m+n} \mid \mathbf{y}_{1:m})$. \square

Hence, the updated mixture of Dirichlet processes is still a mixture of Dirichlet processes with different multiplicities and possibly new point masses in the parameter measures. Iterating the propagation and update operations provided by [Theorem 3.3](#) and [Lemma 3.4](#) yields the computable filter for a partially observed FV process, which sequentially evaluates $\mathcal{L}(X_{t_n} \mid Y_{1:n})$. This is summarised in the next Proposition.

Proposition 3.5. *Consider the family of finite mixtures of Dirichlet processes*

$$\mathcal{F}_{\Pi} = \left\{ \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}} : M \subset \mathcal{M}, |M| < \infty, w_{\mathbf{m}} \geq 0, \sum_{\mathbf{m} \in M} w_{\mathbf{m}} = 1 \right\}.$$

Then, when X_t has generator (10) and data are as in (27), \mathcal{F}_Π is closed under the application of the update and prediction operators (2) and (3). Specifically,

$$(34) \quad \phi_{\mathbf{y}_{m+1:m+n}} \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}} \right) = \sum_{\mathbf{n} \in t(\mathbf{y}_{m+1:m+n}, M)} \hat{w}_{\mathbf{n}} \Pi_{\alpha + \sum_{i=1}^{K_{m+n}} n_i \delta_{y_i^*}},$$

with

$$t(\mathbf{y}, \Lambda) := \{\mathbf{n} : \mathbf{n} = t(\mathbf{y}, \mathbf{m}), \mathbf{m} \in \Lambda\}$$

$$\hat{w}_{\mathbf{n}} \propto w_{\mathbf{m}} \text{PU}_{\alpha}(\mathbf{y}_{m+1:m+n} \mid \mathbf{y}_{1:m}) \quad \text{for } \mathbf{n} = t(\mathbf{y}, \mathbf{m}), \quad \sum_{\mathbf{n} \in t(\mathbf{y}, \Lambda)} \hat{w}_{\mathbf{n}} = 1,$$

and

$$(35) \quad \psi_t \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}} \right) = \sum_{\mathbf{n} \in G(M)} \left(\sum_{\mathbf{m} \in M, \mathbf{m} \geq \mathbf{n}} w_{\mathbf{m}} p_{\mathbf{m}, \mathbf{n}}(t) \right) \Pi_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}}.$$

Proof. The update operation (34) follows directly from Lemma 3.4. The prediction operation (35) for elements of \mathcal{F}_Π follows from Theorem 3.3 together with (26) and a rearrangement of the sums, so that

$$\begin{aligned} \psi_t \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}} \right) &= \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \sum_{\mathbf{n} \in G(\mathbf{m})} p_{\mathbf{m}, \mathbf{n}}(t) \Pi_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}} \\ &= \sum_{\mathbf{n} \in G(M)} \left(\sum_{\mathbf{m} \in M, \mathbf{m} \geq \mathbf{n}} w_{\mathbf{m}} p_{\mathbf{m}, \mathbf{n}}(t) \right) \Pi_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}}. \end{aligned}$$

□

4 Filtering Dawson–Watanabe processes

4.1 The static model: gamma random measures

Gamma random measures can be thought of as the counterpart of Dirichlet processes in the context of finite intensity measures. A gamma random measure $z \in \mathcal{M}(\mathcal{Y})$ with shape parameter α as in (1) and rate parameter $\beta > 0$, denoted $z \sim \Gamma_{\alpha}^{\beta}$, admits representation

$$(36) \quad z(\cdot) = \beta^{-1} \sum_{i=1}^{\infty} Q_i \delta_{Y_i}(\cdot), \quad Y_i \stackrel{iid}{\sim} P_0,$$

with $\{Q_i, i \geq 1\}$ as in (6). The conjugacy and projection properties for gamma random measures are as follows:

- *Conjugacy*: gamma random measures are conjugate with respect to Poisson point processes data. Let N be a Poisson point process on \mathcal{Y} with random intensity measure $z \sim \Gamma_\alpha^\beta$, i.e., conditionally on z , $N(A_i) \stackrel{ind}{\sim} \text{Po}(z(A_i))$ for any disjoint sets $A_1, \dots, A_K \in \mathcal{Y}$, $K \in \mathbb{N}$. Let $m := N(\mathcal{Y})$ and $y_i := N(A_i)/N(\mathcal{Y})$, $i = 1, \dots, K$, so that

$$(37) \quad y_i \mid z, m \stackrel{iid}{\sim} z/|z|, \quad m \mid z \sim \text{Po}(|z|), \quad z \sim \Gamma_\alpha^\beta,$$

where $|z| := z(\mathcal{Y})$ is the total mass of z (recall that $z \in \mathcal{M}(\mathcal{Y})$ is a finite measure). Then

$$(38) \quad z \mid y_{1:m} \sim \Gamma_{\alpha + \sum_{i=1}^m \delta_{y_i}}^{\beta+m}.$$

- *Projection*: for a measurable partition A_1, \dots, A_K of \mathcal{Y} , the vector $(z(A_1), \dots, z(A_K))$ has independent components $z(A_i)$ with gamma distribution $\text{Ga}(\alpha_i, \beta)$, $\alpha_i = \alpha(A_i)$.

The above conjugacy property was showed by [Lo \(1982\)](#). Finally, it is well known that (6) and (36) satisfy the following relation in distribution

$$(39) \quad x(\cdot) = \frac{z(\cdot)}{z(\mathcal{Y})} \sim \Pi_\alpha,$$

where x is independent of $z(\mathcal{Y})$, which extends to the infinite dimensional case the well known relationship between beta and gamma random variables. See for example [Daley and Vere-Jones \(2008\)](#), Example 9.1(e). See also [Feng \(2010\)](#), Section 1.3, for a dynamic analog of (39) linking CIR and WF processes and [Perkins \(1991\)](#) for a measure-valued version.

4.2 The signal: Dawson–Watanabe processes

Dawson–Watanabe (DW) processes are branching measure-valued diffusions taking values in the space of finite discrete measures. See [Dawson \(1993\)](#) and [Li \(2011\)](#) for reviews. Here we are interested in the special case of subcritical branching with immigration, where subcriticality refers to the fact that the mean number of offspring per individual in the underlying population is less than one. We will consider DW processes with generator

$$(40) \quad \mathbb{B}\varphi(z) = \frac{1}{2} \int_{\mathcal{Y}} z(dy) \frac{\partial^2 \varphi(z)}{\partial z(dy)^2} + \int_{\mathcal{Y}} z(dy) \tilde{B} \left(\frac{\partial \varphi(z)}{\partial z(\cdot)} \right) (y),$$

where \tilde{B} is

$$(\tilde{B}f)(y) = \frac{\theta}{2} \int_{\mathcal{Y}} f(u) P_0(dy') - \frac{\beta}{2} f(y), \quad f \in C(\mathcal{Y}),$$

and the domain of \mathbb{B} is as in (10) except that $C^2(\mathbb{R}^k)$ is replaced by its subspace of functions with compact support. Contrary to (10), whose first term describes substitution of individuals in the underlying population, the first term in \mathbb{B} describes addition of individuals through branching, whereas the second accounts for independent immigration of individuals, whose type is selected according to the probability distribution P_0 . These heuristics provide intuition for the fact that DW processes drive evolving measures with non constant mass. The DW process with the above operator is known to be stationary and reversible with respect to the law Γ_α^β of a gamma random measure as in (36) (Shiga, 1990; Ethier and Griffiths, 1993b).

Let Z_t have generator (40). Given a measurable partition A_1, \dots, A_K of \mathcal{Y} , the vector $(Z_t(A_1), \dots, Z_t(A_K))$ has independent components $Z_t(A_i)$ each driven by a Cox–Ingersoll–Ross (CIR) diffusion (Cox, Ingersoll and Ross, 1985). These are also subcritical continuous-state branching processes with immigration, reversible and ergodic with respect to a $\text{Ga}(\alpha_i, \beta)$ distribution, with generator

$$(41) \quad \mathcal{B}_i f(z_i) = \frac{1}{2}(\alpha_i - \beta z_i) f'(z_i) + \frac{1}{2} z_i f''(z_i),$$

acting on $C^2([0, \infty))$ functions which vanish at infinity. See Kawazu and Watanabe (1971).

As for FV and WF processes, a parameter that controls the speed could also be introduced in this case. This corresponds to the original parametrisation by Cox, Ingersoll and Ross (1985), whereby the process has generator

$$\mathcal{C} f(z) = \kappa(\vartheta - z) f'(z) + \frac{\sigma^2}{2} z f''(z),$$

and invariant distribution $\text{Ga}(2\kappa\vartheta/\sigma^2, 2\kappa/\sigma^2)$. Here, for simplicity of exposition, we have set $\alpha = 2\kappa\vartheta/\sigma^2$, $\beta = 2\kappa/\sigma^2$ and $\sigma^2 = 1$.

4.3 Duality and propagation for multivariate CIR signals

Assume the signal $\mathbf{Z}_t = (Z_{1,t}, \dots, Z_{K,t})$ has independent CIR components $Z_{i,t}$ with generator (41). The next proposition identifies the dual process for \mathbf{Z}_t .

Theorem 4.1. *Let $Z_{i,t}$, $i = 1, \dots, K$, be independent CIR processes each with generator (41) parametrised by (α_i, β) , respectively. For $\boldsymbol{\alpha} \in \mathbb{R}_+^K$ and $\theta = |\boldsymbol{\alpha}|$, define $h_{\alpha_i}^C : \mathbb{R}_+ \times \mathbb{Z}_+ \times \mathbb{R}_+$ as*

$$h_{\alpha_i}^C(z, m, s) = \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i + m)} \left(\frac{\beta + s}{\beta} \right)^{\alpha_i} (\beta + s)^m z^m e^{-sz},$$

$h^W : \mathbb{R}_+^K \times \mathbb{Z}_+^K$ as

$$h^W(\mathbf{x}, \mathbf{m}) = \frac{\Gamma(\theta + |\mathbf{m}|)}{\Gamma(\theta)} \prod_{i=1}^K \frac{\Gamma(\alpha_i)}{\Gamma(\alpha_i + m_i)} \mathbf{x}^{\mathbf{m}},$$

and $h : \mathbb{R}_+^K \times \mathbb{Z}_+^K \times \mathbb{R}_+$ as

$$h(\mathbf{z}, \mathbf{m}, s) = h_\theta^C(|\mathbf{z}|, |\mathbf{m}|, s) h^W(\mathbf{x}, \mathbf{m}),$$

where $\mathbf{x} = \mathbf{z}/|\mathbf{z}|$. Then the joint process $\{(Z_{1,t}, \dots, Z_{K,t}), t \geq 0\}$ is dual, in the sense of (4), to the process $\{(\mathbf{M}_t, S_t), t \geq 0\} \subset \mathbb{Z}_+^K \times \mathbb{R}_+$ with generator

$$(42) \quad Bg(\mathbf{m}, s) = \frac{1}{2}|\mathbf{m}|(\beta + s) \sum_{i=1}^K \frac{m_i}{|\mathbf{m}|} [g(\mathbf{m} - \mathbf{e}_i, s) - g(\mathbf{m}, s)] - \frac{1}{2}s(\beta + s) \frac{\partial g(\mathbf{m}, s)}{\partial s}$$

with respect to $h(\mathbf{z}, \mathbf{m}, s)$.

Proof. Throughout the proof, for ease of notation we will write h_i^C instead of $h_{\alpha_i}^C$. Note first that for all $\mathbf{m} \in \mathbb{Z}_+^K$ we have

$$(43) \quad \prod_{i=1}^K h_i^C(z_i, m_i, s) = h_\theta^C(|z|, |\mathbf{m}|, s) h^W(\mathbf{x}, \mathbf{m}),$$

where $x_i = z_i/|z|$, which follows from direct computation by multiplying and dividing by the correct ratios of gamma functions and by writing $\prod_{i=1}^K z_i^{m_i} = |z|^m \prod_{i=1}^K x_i^{m_i}$. We show the result for $K = 2$, from which the statement for general K case follows easily. From the independence of the CIR processes, the generator $(Z_{1,t}, Z_{2,t})$ applied to the left hand side of (43) is

$$(44) \quad (\mathcal{B}_1 + \mathcal{B}_2)h_1^C h_2^C = h_2^C \mathcal{B}_1 h_1^C + h_1^C \mathcal{B}_2 h_2^C.$$

A direct computation shows that

$$\begin{aligned} \mathcal{B}_i h_i^C &= \frac{m_i}{2}(\beta + s) h_i^C(z_i, m_i - 1, s) + \frac{s}{2}(\alpha_i + m_i) h_i^C(z_i, m_i + 1, s) \\ &\quad - \frac{1}{2}(s(\alpha_i + m_i) + m_i(\beta + s)) h_i^C(z_i, m_i, s). \end{aligned}$$

Substituting in the right hand side of (44) and collecting terms with the same coefficients gives

$$\begin{aligned} &\frac{\beta + s}{2} \left[m_1 h_1^C(z_1, m_1 - 1, s) h_2^C(z_2, m_2, s) + m_2 h_1^C(z_1, m_1, s) h_2^C(z_2, m_2 - 1, s) \right] \\ &+ \frac{s}{2} \left[(\alpha_1 + m_1) h_1^C(z_1, m_1 + 1, s) h_2^C(z_2, m_2, s) + (\alpha_2 + m_2) h_1^C(z_1, m_1, s) h_2^C(z_2, m_2 + 1, s) \right] \\ &- \frac{1}{2} (s(\alpha + m) + m(\beta + s)) h_1^C(z_1, m_1, s) h_2^C(z_2, m_2, s) \end{aligned}$$

with $\alpha = \alpha_1 + \alpha_2$ and $m = m_1 + m_2$. From (43) we now have

$$\begin{aligned} & \frac{\beta + s}{2} h_\theta^C(|z|, m - 1, s) \left[m_1 h^W(\mathbf{x}, \mathbf{m} - \mathbf{e}_1, s) + m_2 h^W(\mathbf{x}, \mathbf{m} - \mathbf{e}_2, s) \right] \\ & + \frac{s}{2} h_\theta^C(|z|, m + 1, s) \left[(\alpha_1 + m_1) h^W(\mathbf{x}, \mathbf{m} + \mathbf{e}_1, s) + (\alpha_2 + m_2) h^W(\mathbf{x}, \mathbf{m} + \mathbf{e}_2, s) \right] \\ & - \frac{1}{2} (s(\alpha + m) + m(\beta + s)) h_\theta^C(|z|, m, s) h^W(\mathbf{x}, \mathbf{m}, s). \end{aligned}$$

Then

$$\begin{aligned} (\mathcal{B}_1 + \mathcal{B}_2) h_1^C h_2^C &= \frac{\beta + s}{2} \left[m_1 h(\mathbf{z}, \mathbf{m} - \mathbf{e}_1, s) + m_2 h(\mathbf{z}, \mathbf{m} - \mathbf{e}_2, s) \right] \\ (45) \quad & + \frac{s}{2} \left[(\alpha_1 + m_1) h(\mathbf{z}, \mathbf{m} + \mathbf{e}_1, s) + (\alpha_2 + m_2) h(\mathbf{z}, \mathbf{m} + \mathbf{e}_2, s) \right] \\ & - \frac{1}{2} (s(\alpha + m) + m(\beta + s)) h(\mathbf{z}, \mathbf{m}, s). \end{aligned}$$

Noting now that

$$\frac{\partial}{\partial s} h(\mathbf{z}, \mathbf{m}, s) = \frac{\alpha + m}{\beta + s} h(\mathbf{z}, \mathbf{m}, s) - \frac{\alpha_1 + m_1}{\beta + s} h(\mathbf{z}, \mathbf{m} + \mathbf{e}_1, s) - \frac{\alpha_2 + m_2}{\beta + s} h(\mathbf{z}, \mathbf{m} + \mathbf{e}_2, s),$$

an application of (42) on $h(\mathbf{z}, \mathbf{m}, s)$ shows that $(Bh(\mathbf{z}, \cdot, \cdot))(\mathbf{m}, s)$ equals the right hand side of (45), so that (5) holds, giving the result. \square

The previous Theorem extends the gamma-type duality showed for one dimensional CIR processes in Papaspiliopoulos and Ruggiero (2014). Here \mathbf{M}_t is a K -dimensional death process $\mathbf{M}_t \subset \mathbb{Z}_+^K$ which, conditionally on S_t , jumps from \mathbf{m} to $\mathbf{m} - \mathbf{e}_i$ at rate $2m_i(\beta + S_t)$, and $S_t \in \mathbb{R}_+$ is a nonnegative deterministic process driven by the logistic type differential equation

$$(46) \quad \frac{dS_t}{dt} = -\frac{1}{2} S_t (\beta + S_t).$$

The next Proposition formalises the propagation step for multivariate CIR processes. Denote by $\mathbf{Ga}(\boldsymbol{\alpha}, \beta)$ the product of gamma distributions $\text{Ga}(\alpha_1, \beta) \times \cdots \times \text{Ga}(\alpha_K, \beta)$, with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_K)$.

Proposition 4.2. *Let $\{(Z_{1,t}, \dots, Z_{K,t}), t \geq 0\}$ be as in Theorem 4.1. Then*

$$\begin{aligned} (47) \quad \psi_t(\mathbf{Ga}(\boldsymbol{\alpha} + \mathbf{m}, \beta + s)) &= \\ &= \sum_{i=0}^{|\mathbf{m}|} \text{Bin}(|\mathbf{m}| - i; |\mathbf{m}|, p(t)) \text{Ga}(\boldsymbol{\theta} + |\mathbf{m}| - i, \beta + S_t) \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{m}, |\mathbf{i}|=i} p(\mathbf{i}; \mathbf{m}, i) \pi_{\boldsymbol{\alpha} + \mathbf{m} - \mathbf{i}}, \end{aligned}$$

where

$$(48) \quad p(t) = \frac{\beta}{(\beta + s)e^{\beta t/2} - s}, \quad S_t = \frac{\beta s}{(\beta + s)e^{\beta t/2} - s}, \quad S_0 = s$$

and $p(\mathbf{i}; \mathbf{m}, i)$ is as in (17).

Proof. From independence we have

$$\psi_t(\mathbf{Ga}(\boldsymbol{\alpha} + \mathbf{m}, \beta + s)) = \prod_{i=1}^K \psi_t(\text{Ga}(\alpha_i + m_i, \beta + s)).$$

Using Lemma A.2 in the Appendix, the previous equals

$$\begin{aligned} & \prod_{i=1}^K \sum_{j=0}^{m_i} \text{Bin}(m_i - j; m_i, p(t)) \text{Ga}(\alpha_i + m_i - j, \beta + S_t) \\ &= \sum_{i_1=0}^{m_1} \text{Bin}(m_1 - i_1; m_1, p(t)) \text{Ga}(\alpha_1 + m_1 - i_1, \beta + S_t) \\ & \quad \times \cdots \times \sum_{i_K=0}^{m_K} \text{Bin}(m_K - i_K; m_K, p(t)) \text{Ga}(\alpha_K + m_K - i_K, \beta + S_t). \end{aligned}$$

Using now the fact that a product of Binomials equals the product of a Binomial and an hypergeometric distribution, we have

$$\sum_{i=0}^{|\mathbf{m}|} \text{Bin}(|\mathbf{m}| - i; |\mathbf{m}|, p(t)) \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{m}, |\mathbf{i}|=i} p(\mathbf{i}; \mathbf{m}, i) \prod_{j=1}^K \text{Ga}(\alpha_j + m_j - i_j, \beta + S_t)$$

which, using (39), yields (47). Furthermore, (48) is obtained by solving (46) and by means of the following argument. The one dimensional death process that drives $|\mathbf{M}_t|$ in Theorem 4.1, jumps from $|\mathbf{m}|$ to $|\mathbf{m}| - 1$ at rate $|\mathbf{m}|(\beta + S_t)/2$, see (42). Hence the probability of this event not occurring in $[0, t]$ is

$$P(|\mathbf{m}| \mid |\mathbf{m}|, S_t) = \exp \left\{ -\frac{|\mathbf{m}|}{2} \int_0^t (\beta + S_u) du \right\} = \left(\frac{\beta}{(\beta + s)e^{\beta t/2} - s} \right)^{|\mathbf{m}|}.$$

The probability of a jump from $|\mathbf{m}|$ to $|\mathbf{m}| - 1$ occurring in $[0, t]$ is

$$P(|\mathbf{m}| - 1 \mid |\mathbf{m}|, S_t)$$

$$\begin{aligned}
&= \int_0^t \exp \left\{ -\frac{|\mathbf{m}|}{2} \int_0^s (\beta + S_u) du \right\} \frac{|\mathbf{m}|}{2} S_s \exp \left\{ -\frac{|\mathbf{m}|-1}{2} \int_s^t (\beta + S_u) du \right\} dt \\
&= \frac{|\mathbf{m}|}{2} \exp \left\{ -\frac{|\mathbf{m}|}{2} \int_0^t (\beta + S_u) du \right\} \int_0^t S_s \exp \left\{ \left(\frac{|\mathbf{m}|}{2} - \frac{|\mathbf{m}|-1}{2} \right) \int_s^t (\beta + S_u) du \right\} dt \\
&= |\mathbf{m}| \exp \left\{ -\frac{|\mathbf{m}|}{2} \int_0^t (\beta + S_u) du \right\} \left(1 - \exp \left\{ \left(\frac{|\mathbf{m}|}{2} - \frac{|\mathbf{m}|-1}{2} \right) \int_0^t (\beta + S_u) du \right\} \right) \\
&= |\mathbf{m}| \left(\exp \left\{ -\frac{|\mathbf{m}|}{2} \int_0^t (\beta + S_u) du \right\} - \exp \left\{ -\frac{|\mathbf{m}|-1}{2} \int_0^t (\beta + S_u) du \right\} \right) \\
&= |\mathbf{m}| \left(\frac{\beta}{(\beta + s)e^{\beta t/2} - s} \right)^{|\mathbf{m}|-1} \left(1 - \frac{\beta}{(\beta + s)e^{\beta t/2} - s} \right).
\end{aligned}$$

Iterating the argument leads to conclude that the death process jumps from $|\mathbf{m}|$ to $|\mathbf{m}| - i$ in $[0, t]$ with probability $\text{Bin}(|\mathbf{m}| - i \mid |\mathbf{m}|, p(t))$. \square

Note that when $s \in \mathbb{N}$, $\text{Ga}(\alpha_i + m, \beta + s)$ is the posterior distribution of a parameter with $\text{Ga}(\alpha_i, \beta)$ distribution, given s Poisson observations whose sum is m . Hence the dual component $M_{i,t}$ is interpreted as the sum of the observed values of type i , and $S_t \subset \mathbb{R}_+$ as a continuous version of the sample size.

4.4 Filtering Dawson–Watanabe processes

Let now the signal Z_t follow a DW process with generator (40), and let Γ_α^β be the law of a gamma random measure, defined in (36). The following result provides the propagation step for filtering the DW process.

Theorem 4.3. *Let $p(t)$ and S_t be as in (48) and $p(\mathbf{n}; \mathbf{m}, |\mathbf{n}|)$ as in (17). Then*

$$(49) \quad \psi_t \left(\Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta + s} \right) = \sum_{\mathbf{n} \in G(\mathbf{m})} \tilde{p}_{\mathbf{m}, \mathbf{n}}(t) \Gamma_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}}^{\beta + S_t},$$

where

$$(50) \quad \tilde{p}_{\mathbf{m}, \mathbf{n}}(t) = \text{Bin}(|\mathbf{m}| - |\mathbf{n}| \mid |\mathbf{m}|, p(t)) p(\mathbf{n}; \mathbf{m}, |\mathbf{n}|),$$

and $G(M)$ is as in (21).

Proof. Fix a partition (A_1, \dots, A_K) of \mathcal{Y} . Then by Proposition 4.2

$$\psi_t \left(\Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta + s} (A_1, \dots, A_K) \right)$$

$$= \sum_{i=0}^{|\mathbf{m}|} \text{Bin}(|\mathbf{m}| - i; |\mathbf{m}|, p(t)) \text{Ga}(\theta + |\mathbf{m}| - i, \beta + S_t) \sum_{\mathbf{0} \leq \mathbf{i} \leq \tilde{\mathbf{m}}, |\mathbf{i}|=i} p(\mathbf{i}; \tilde{\mathbf{m}}, i) \pi_{\alpha + \tilde{\mathbf{m}} - \mathbf{i}},$$

where $\Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s}(A_1, \dots, A_K)$ denotes $\Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s}(\cdot)$ evaluated on (A_1, \dots, A_K) and $\tilde{\mathbf{m}}$ are the multiplicities yielded by the projection of \mathbf{m} onto (A_1, \dots, A_K) . Use now (39) and (50) to write the right hand side of (49) as

$$\begin{aligned} & \sum_{\mathbf{n} \in G(\mathbf{m})} \tilde{p}_{\mathbf{m}, \mathbf{n}}(t) \Gamma_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}}^{\beta+S_t} \\ &= \sum_{i=0}^{|\mathbf{m}|} \text{Bin}(|\mathbf{m}| - i; |\mathbf{m}|, p(t)) \text{Ga}(\theta + |\mathbf{m}| - i, \beta + S_t) \sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{m}, |\mathbf{n}|=i} p(\mathbf{n}; \mathbf{m}, i) \Pi_{\alpha + \sum_{j=1}^{K_m} (m_j - n_j) \delta_{y_j^*}}. \end{aligned}$$

Since the inner sum is the only term which depends on multiplicities and Since Dirichlet processes are characterised by their finite-dimensional projections, we are only left to show that

$$\sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{m}, |\mathbf{n}|=i} p(\mathbf{n}; \mathbf{m}, i) \Pi_{\alpha + \sum_{j=1}^{K_m} (m_j - n_j) \delta_{y_j^*}}(A_1, \dots, A_K) = \sum_{\mathbf{0} \leq \mathbf{i} \leq \tilde{\mathbf{m}}, |\mathbf{i}|=i} p(\mathbf{i}; \tilde{\mathbf{m}}, i) \pi_{\alpha + \tilde{\mathbf{m}} - \mathbf{i}}$$

which, in view of (29), holds if

$$\sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{m}: \tilde{\mathbf{n}} = \mathbf{i}} p(\mathbf{i}; \mathbf{m}, i) = p(\mathbf{i}; \tilde{\mathbf{m}}, i),$$

where $\tilde{\mathbf{n}}$ denotes the projection of \mathbf{n} onto (A_1, \dots, A_K) . This is the consistency with respect to merging of classes of the multivariate hypergeometric distribution, and so the result now follows by the same argument at the end of the proof of Theorem 3.3. \square

The next proposition provides the update step for mixtures of gamma priors.

Proposition 4.4. *Let $y_{m+i} \mid z \sim z$, $i = 1, \dots, n$, with*

$$z \sim \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s}, \quad \sum_{\mathbf{m} \in M} w_{\mathbf{m}} = 1.$$

Then

$$(51) \quad \phi_{\mathbf{y}_{m+1:m+n}} \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s} \right) = \sum_{\mathbf{n} \in t(\mathbf{y}_{m+1:m+n}, M)} \hat{w}_{\mathbf{n}} \Gamma_{\alpha + \sum_{i=1}^{K_{m+n}} n_i \delta_{y_i^*}}^{\beta+s+n-m},$$

with $t(\cdot)$ is as in Proposition 3.2 and $\hat{w}_{\mathbf{n}}$ as in (33).

Proof. Since $z_{\mathbf{m}} := (z \mid \mathbf{m}) \sim \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s}$, from (37) we have

$$y_{m+1}, \dots, y_n \mid z, \mathbf{m}, n \stackrel{iid}{\sim} z_{\mathbf{m}} / |z_{\mathbf{m}}|, \quad n \mid z_{\mathbf{m}} \sim \text{Po}(|z_{\mathbf{m}}|).$$

Using (39) we have

$$\Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s} = \text{Ga}(\theta + |\mathbf{m}|, \beta + s) \Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}},$$

that is $|z_{\mathbf{m}}|$ and $z_{\mathbf{m}}/|z_{\mathbf{m}}|$ are independent with $\text{Ga}(\theta + |\mathbf{m}|, \beta + s)$ and $\Pi_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}$ distribution respectively. (38) now implies that

$$z_{\mathbf{m}} \mid \mathbf{y}_{m+1:m+n} \sim \text{Ga}(\theta + |\mathbf{n}|, \beta + s + n - m) \Pi_{\alpha + \sum_{i=1}^{K_{m+n}} n_i \delta_{y_i^*}} = \Gamma_{\alpha + \sum_{i=1}^{K_{m+n}} n_i \delta_{y_i^*}}^{\beta+s+n-m}$$

where \mathbf{n} are the multiplicities of the distinct values in $\mathbf{y}_{1:n}$. Finally, by the independence of $|z_{\mathbf{m}}|$ and $z_{\mathbf{m}}/|z_{\mathbf{m}}|$, the conditional distribution of the mixing measure follows by the same argument used in Proposition 3.4. \square

The successive iteration of the update and prediction step given by Theorem 4.3 and Proposition 4.4 provides the filter for DW signals with data as in (37). This is summarised in the next Proposition.

Proposition 4.5. *Consider the family of finite mixtures of gamma random measures*

$$\mathcal{F}_{\Gamma} = \left\{ \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s} : s > 0, M \subset \mathcal{M}, |M| < \infty, w_{\mathbf{m}} \geq 0, \sum_{\mathbf{m} \in M} w_{\mathbf{m}} = 1 \right\},$$

with \mathcal{M} as in (20). Then, when Z_t has generator (40) and data are as in (37), \mathcal{F}_{Γ} is closed under the application of the update and prediction operators (2) and (3). Specifically,

$$(52) \quad \phi_{\mathbf{y}_{m+1:m+n}} \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s} \right) = \sum_{\mathbf{n} \in t(\mathbf{y}_{m+1:m+n}, M)} \hat{w}_{\mathbf{n}} \Gamma_{\alpha + \sum_{i=1}^{K_{m+n}} n_i \delta_{y_i^*}}^{\beta+s+n-m},$$

with

$$t(\mathbf{y}, \Lambda) := \{\mathbf{n} : \mathbf{n} = t(\mathbf{y}, \mathbf{m}), \mathbf{m} \in \Lambda\}$$

$$\hat{w}_{\mathbf{n}} \propto w_{\mathbf{m}} \text{PU}_{\alpha}(\mathbf{y}_{m+1:m+n} \mid \mathbf{y}_{1:m}) \quad \text{for } \mathbf{n} = t(\mathbf{y}, \mathbf{m}), \quad \sum_{\mathbf{n} \in t(\mathbf{y}, \Lambda)} \hat{w}_{\mathbf{n}} = 1,$$

and

$$\psi_t \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s} \right) = \sum_{\mathbf{n} \in G(M)} \left(\sum_{\mathbf{m} \in M, \mathbf{m} \geq \mathbf{n}} w_{\mathbf{m}} \tilde{p}_{\mathbf{m}, \mathbf{n}}(t) \right) \Gamma_{\alpha + \sum_{i=1}^{K_m} n_i \delta_{y_i^*}}^{\beta+S_t}.$$

Proof. The update operation (52) follows directly from Lemma 3.4. The prediction operation (35) for elements of \mathcal{F}_Π follows from Theorem 4.3 together with (26) and a rearrangement of the sums, so that

$$\begin{aligned} \psi_t \left(\sum_{\mathbf{m} \in M} w_{\mathbf{m}} \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+s} \right) &= \sum_{\mathbf{m} \in M} w_{\mathbf{m}} \sum_{\mathbf{n} \in G(\mathbf{m})} p_{\mathbf{m},\mathbf{n}}(t) \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+S_t} \\ &= \sum_{\mathbf{n} \in G(M)} \left(\sum_{\mathbf{m} \in M, \mathbf{m} \geq \mathbf{n}} w_{\mathbf{m}} p_{\mathbf{m},\mathbf{n}}(t) \right) \Gamma_{\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}}^{\beta+S_t}. \end{aligned}$$

□

5 Interpretations and concluding remarks

We have derived explicit filters that allow sequential evaluation of the marginal posterior distributions of hidden FV and DW signals given appropriate observations. The algorithms provided in Propositions 3.5 and 4.5, which extend previous results on conjugacy of mixtures of Dirichlet and gamma priors (Antoniak, 1974; Lo, 1982), can be interpreted as follows. Without observations, our knowledge of the signal amounts to the prior distribution, i.e., the law of the FV or DW process, which provides the instant-wise information encoded in the stationary distributions Π_α and Γ_α^β respectively. When new observations become available, this new information is integrated into our current knowledge by means of the update operator, which modifies the prior parameters according to the atoms observed in the sample, yielding $\beta+m$ for the DW case and $\alpha + \sum_{i=1}^{K_m} m_i \delta_{y_i^*}$ for both. In the following interval of time, before further observations become available, the distribution of the signals becomes a mixture that gradually approaches the ergodic distribution. This implies that the signal progressively forgets the previously acquired information, which becomes obsolete and gradually swamped by the prior. This mechanism is enforced by a time-continuous modification of the mixture weights, which modulate the sequential random removal of the atoms previously added to the base measure, governed by the death process. In addition, in the DW case a deterministic process governs the sample size parameter, which increases by jumps in the update step but decreases continuously in the propagation, gradually bringing it back to its ergodic state. The dual process, which is related to the time reversal structure of the signal, is thus able to isolate the prior knowledge on the signal from the posterior information acquired with the observations, and dictates how this information is to be dealt with for filtering.

For what concerns the strategy followed for proving the propagation result in Theorems 3.3 and 4.3, one could be tempted to work directly with the duals of the FV and DW processes

(see Dawson and Hochberg, 1982; Ethier and Kurtz, 1993; Etheridge, 2000). However, this is not optimal, due to the high degree of generality of such dual processes. The simplest path for deriving the propagation step for the nonparametric signals appears to be resorting to the corresponding parametric dual by means of projections and by exploiting the filtering results for those cases.

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Appendix

The following Lemma provides the transition probabilities for the death process of Theorem 3.1.

Lemma A.1. *Let $M_t \subset \mathbb{Z}_+^\infty$ be a death process that starts from $M_0 = \mathbf{m}_0 \in \mathcal{M}$ (see (20)) and jumps from \mathbf{m} to $\mathbf{m} - \mathbf{e}_i$ at rate $m_i(\theta + |\mathbf{m}| - 1)/2$, with generator*

$$\frac{\theta + |\mathbf{m}| - 1}{2} \sum_{i \geq 1} m_i h(\mathbf{x}, \mathbf{m} - \mathbf{e}_i) - \frac{|\mathbf{m}|(\theta + |\mathbf{m}| - 1)}{2} h(\mathbf{x}, \mathbf{m}).$$

Then the transition probabilities for M_t are

$$\begin{aligned} p_{\mathbf{m}, \mathbf{m}}(t) &= e^{-\lambda_{|\mathbf{m}|}}, \\ p_{\mathbf{m}, \mathbf{m} - \mathbf{i}}(t) &= C_{|\mathbf{m}|, |\mathbf{m}| - |\mathbf{i}|}(t) \left(\prod_{h=0}^{|\mathbf{i}| - 1} \lambda_{|\mathbf{m}| - h} \right) p(\mathbf{i}; \mathbf{m}, |\mathbf{i}|), \quad \mathbf{0} < \mathbf{i} \leq \mathbf{m}, \\ C_{|\mathbf{m}|, |\mathbf{m}| - |\mathbf{i}|}(t) &= (-1)^{|\mathbf{i}|} \sum_{k=0}^{|\mathbf{i}|} \frac{e^{-\lambda_{|\mathbf{m}| - k} t}}{\prod_{0 \leq h \leq |\mathbf{i}|, h \neq k} (\lambda_{|\mathbf{m}| - k} - \lambda_{|\mathbf{m}| - h})}, \end{aligned}$$

and 0 otherwise.

Proof. Since $|\mathbf{m}_0| < \infty$, for any such \mathbf{m}_0 the proof is analogous to that of Proposition 2.1 in Papaspiliopoulos and Ruggiero (2014). \square

The following Lemma recalls the propagation step for one dimensional CIR processes.

Lemma A.2. *Let $Z_{i,t}$ be a CIR process with generator (41) and invariant distribution $\text{Ga}(\alpha_i, \beta)$. Then*

$$\psi_t(\text{Ga}(\alpha_i + m, \beta + s)) = \sum_{j=0}^m \text{Bin}(m - j; m, p(t)) \text{Ga}(\alpha_i + m - j, \beta + S_t),$$

where

$$p(t) = \frac{\beta}{(\beta + s)e^{\beta t/2} - s}, \quad S_t = \frac{\beta s}{(s + \beta)e^{\beta t/2} - s}, \quad S_0 = s.$$

Proof. It follows from Section 3.1 in Papaspiliopoulos and Ruggiero (2014) by letting $\alpha = \delta/2$, $\beta = \gamma/\sigma^2$ and $S_t = \Theta_t - \beta$. \square

References

- ANTONIAK, C.E. (1974). Mixtures of Dirichlet processes with applications to Bayesian non-parametric problems. *Ann. Statist.* **2**, 1152–1174.
- BEAL, M.J., GHAHRAMANI, Z. and RASMUSSEN, C.E. (2002). The infinite hidden Markov model. *Advances in Neural Information Processing Systems* **14**, 577–585.
- BLACKWELL, D. and MACQUEEN, J.B. (1973). Ferguson distributions via Pólya urn schemes. *Ann. Statist.* **1**, 353–355.
- COX, J.C., INGERSOLL, J.E. and ROSS, S.A. (1985). A theory of the term structure of interest rates. *Econometrica* **53**, 385–407.
- CAPPÉ, O., MOULINES, E. and RYDÉN, T. (2005). *Inference in hidden Markov models*. Springer.
- CHALEYAT-MAUREL, M. and GENON-CATALOT, V. (2006). Computable infinite-dimensional filters with applications to discretized diffusion processes. *Stoch. Proc. Appl.* **116**, 1447–1467.
- CHALEYAT-MAUREL, M. and GENON-CATALOT, V. (2009). Filtering the Wright–Fisher diffusion. *ESAIM Probab. Stat.* **13**, 197–217.
- DALEY, D.J. and VERE-JONES (2008). *An introduction to the theory of point processes, Vol. 2*. Springer, New York.
- DAWSON, D.A. (1993). *Measure-valued Markov processes*. Ecole d’Eté de Probabilités de Saint Flour XXI. Lecture Notes in Mathematics **1541**. Springer, Berlin.
- DAWSON, D.A. (2010). *Introductory lectures on stochastic population systems*. Technical Report Series **451**, Laboratory for Research in Statistics and Probability, Carleton University.
- DAWSON, D.A. and HOCHBERG, K.J. (1982). Wandering random measures in the Fleming–Viot model. *Ann. Probab.* **10**, 554–580.
- ETHERIDGE, A.M. (2000). *An introduction to superprocesses*. University Lecture Series, 20. American Mathematical Society, Providence, RI.

- ETHIER, S.N. (1981). A class of infinite-dimensional diffusions occurring in population genetics. *Indiana Univ. Math. J.* **30**, 925–935.
- ETHIER, S.N. and GRIFFITHS, R.C. (1993). The transition function of a Fleming–Viot process. *Ann. Probab.* **21**, 1571–1590.
- ETHIER, S.N. and GRIFFITHS, R.C. (1993b). The transition function of a measure-valued branching diffusion with immigration. In *Stochastic Processes. A Festschrift in Honour of Gopinath Kallianpur* (S. Cambanis, J. Ghosh, R.L. Karandikar and P.K. Sen, eds.), 71–79. Springer, New York.
- ETHIER, S.N. and KURTZ, T.G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optim.* **31**, 345–386.
- FENG, S. (2010). *The Poisson–Dirichlet distribution and related topics*. Springer, Heidelberg.
- FERGUSON, T.S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.* **1**, 209–230.
- GENON-CATALOT, V. and KESSLER, M. (2004). Random scale perturbation of an AR(1) process and its properties as a nonlinear explicit filter. *Bernoulli*, **10**, 701–720.
- GHOSAL, S. (2010). The Dirichlet process, related priors and posterior asymptotics. In *Bayesian Nonparametrics* (N. L. Hjort, C. C. Holmes, P. Müller and S. G. Walker, eds.). Cambridge Univ. Press, Cambridge.
- GRIFFIN, J. E. and STEEL, M. F. J. (2006). Order-based dependent Dirichlet processes. *JASA*, **473**, 179–194.
- JOHNSON, N.L., KOTZ, S. and BALAKRISHNAN, N. (1997). *Discrete multivariate distributions*. John Wiley & Sons, New York.
- KAWAZU, K. and WATANABE, S. (1971). Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16**, 36–54.
- LI, Z. (2011). *Measure-valued branching Markov processes*. Springer, Heidelberg.
- LO, A.Y. (1982). Bayesian nonparametric statistical inference for Poisson point process. *Z. Wahrsch. Verw. Gebiete* **59**, 55–66.
- MENA, R.H. and RUGGIERO, M. (2014). Dynamic density estimation with diffusive Dirichlet mixtures. *Bernoulli*, in press.
- PAPASPILOPOULOS, O. and RUGGIERO, M. (2014). Optimal filtering and the dual process. *Bernoulli* **20**, 1999–2019.
- PERKINS, E.A. (1991). Conditional Dawson–Watanabe processes and Fleming–Viot processes. In *Seminar on Stochastic processes*, 143–156, Birkhäuser, Boston.
- RODRIGUEZ, A. AND TER HORST, E. (2008). Bayesian dynamic density estimation. *Bayes. Anal.* **3**, 339–366.
- STEPLETON, T., GHAHRAMANI, Z., GORDON, G., and LEE, T.-S. (2009). The block diagonal

- infinite hidden Markov model. *Journal of Machine Learning Research* **5**, 544–551.
- SETHURAMAN, J. (1994). A constructive definition of the Dirichlet process prior. *Statist. Sinica* **2**, 639–650.
- SHIGA, T. (1990). A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes. *J. Math. Kyoto Univ.* **30**, 245–279.
- VAN GAEL, V., SAATCI, Y., TEH, Y.W. and GHAHRAMANI, Z. (2008). Beam sampling for the infinite hidden Markov model. In *Proceedings of the 25th international conference on Machine learning*.
- YAU, C., PAPASPILIOPOULOS, O., ROBERTS, G.O. and HOLMES, C. (2011). Bayesian non-parametric hidden Markov models with applications in genomics *JRSS, series B* **73**, 37–57.
- ZHANG, A., ZHU, J. and ZHANG, B. (2014). Max-margin infinite hidden Markov models. In *Proceedings of the 31st international conference on Machine learning*.